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Hence, three particular integrals of the equation, with second member 0, are

$$y_1 = x^2, \qquad y_2 = \frac{x^2}{a - bx}, \qquad y_3 = \frac{x^3}{a - bx},$$

only two of which are independent.

We may now put $y = y_1 v$, or $y = y_2 w$, or $y = y_3 z$, etc.: or $(\theta - 2)y = v$, or[1] $[a(\theta - 2) - bx(\theta - 1)]y = w$, or $[a(\theta - 3) - bx(\theta - 2)]y = z$, etc.

The substitution $y = y_2 w$ leads to the *normal* form, which could have been obtained in the ordinary way,

$$\frac{d^2w}{dx^2} = \frac{6a^2}{x^4}.$$

Hence,

$$w = A + Bx + \frac{a^2}{x^2}.$$

$$\therefore \quad y = \frac{x^2}{a - bx}\left[A + Bx + \frac{a^2}{x^2}\right].$$

Again, the equation may be written

$$[(\theta - 2)(\theta - 3)(a - bx)]y = 6a^2.$$

$$\therefore \quad (a - bx)y = Ax^2 + Bx^3 + a^2.$$

Also solved by V. M. SPUNAR, ELIJAH SWIFT, A. M. HARDING, and ELMER SCHUYLER.

376. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that

$$\frac{1}{z} - \frac{1}{z}(1 - 2xz + z^2)^{1/2} = x + \frac{z}{2}\left(\frac{x^2 - 1}{1 - xz}\right) + \sum_{n=2}^{n=\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n - 3)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}(x^2 - 1)^n\left(\frac{z}{1 - xz}\right)^{2n-1}$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

First of all the coefficients of the terms of the infinite series are the same as those in the expansion of $-(1 - a)^{\frac{1}{2}}$. These lead us to expand

$$\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2}$$

by the Binominal theorem, which gives

$$1 - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right)^2 - \left\{\frac{1\cdot 1}{2\cdot 4}(x^2 - 1)^2\left(\frac{z}{1 - xz}\right)^4 + \cdots\right\}.$$

Noting that the terms in the bracket are the corresponding terms in the infinite series each multiplied by $z/(1 - xz)$, we find that

$$\sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n - 3)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}(x^2 - 1^n)\left(\frac{z}{1 - xz}\right)^{2n-1}$$
$$= -\frac{1 - xz}{z}\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2} + \frac{1 - xz}{z} - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right).$$

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[1] These two forms of solution are indicated in the German Edition of Forsyth.

Substituting this value, the result follows by a simple algebraic reduction. The result is true only when the series converges, *i. e.*, when $|(x^2-1)(z/(1-xz))^2|<1$ or $x<(1+z^2)/2z$.

Also solved by S. A. Joffe, Geo. W. Hartwell, Albert N. Nauer, and A. M. Harding.
Note.—This problem should have been placed in the Algebra department. Editors.

377. Proposed by W. D. CAIRNS, Oberlin College.

It is required to find a curve of the form $y=x(x-a)(x-b)$ such that the abscissas of the maximum and minimum values, as well as a and b, shall be positive integers.

Solution by Ralph D. Beetle, Dartmouth College.

The abscissas of the minimum and maximum values are

$$(1)\qquad x_1=\tfrac{1}{3}[a+b+\sqrt{a^2-ab+b^2}],\qquad x_2=\tfrac{1}{3}[a+b-\sqrt{a^2-ab+b^2}].$$

If a and b are positive integers, so also are x_1 and x_2 if, and only if, $a+b$ is divisible by 3 and a^2-ab+b^2 is a perfect square divisible by 9. These conditions are equivalent to the simpler ones that a and b be divisible by 3 and a^2-ab+b^2 be a perfect square. If we write

$$(2)\qquad a=3h,\ b=3k$$

our problem is reduced to the finding of positive integral values of h and k such that h^2-hk+k^2 is a perfect square. We note at once that, if a and b are equal, it is sufficient that they be multiples of 3. We may find all solutions of the problem as follows:

Without loss of generality, we may assume that $h\leqq k$, so that $h^2-hk+k^2\leqq k^2$. If we put $h^2-hk+k^2=(k-r)^2$, it is found that

$$(3)\qquad h=\tfrac{1}{2}[k\pm\sqrt{k^2-8rk+4r^2}].$$

In order that h be an integer, we must have $k^2-8rk+4r^2=t^2$, where t is an integer, or $(k-4r)^2-t^2=12r^2$. It is evident that k and t are both even or both odd, and we may therefore put

$$(4)\qquad k-4r=n+s,\qquad t=n-s,$$

where n and s are positive integers. Then

$$(5)\qquad ns=3r^2,$$

and the values of a, b, x_1, x_2 in terms of r, n, s are found from (1), (2), (3), (4) to be either

$$a=3n+6r\qquad b=3n+12r+3s,\qquad x_1=3n+9r+2s,\qquad x_2=n+3r,$$

or

$$a=6r+3s,\qquad b=3n+12r+3s,\qquad x_1=2n+9r+3s,\qquad x_2=3r+s.$$